

# A Categorification of the Spin Representation of $U(\mathfrak{so}(7, \mathbb{C}))$ via Projective Functors

Yongjun Xu, Shilin Yang<sup>†</sup>

College of Applied Sciences, Beijing University of Technology  
Beijing 100124, P. R. China

**Abstract.** The purpose of this paper is to study a categorification of the  $n$ -th tensor power of the spin representation of  $U(\mathfrak{so}(7, \mathbb{C}))$  by using certain subcategories and projective functors of the BGG category of the complex Lie algebra  $\mathfrak{gl}_n$ .

**Key Words:** BGG category; Categorification; Lie algebra; Projective functor; Spin representation.

**Mathematics Subject Classification:** 17B10.

## 1. INTRODUCTION

The general idea of categorification was introduced by Crane and Frenkel [4, 5]. In recent years categorifications of algebras and their representations have been studied by many mathematicians, see for example [10, 13, 14, 18] and references therein.

Let  $\mathcal{O}(\mathfrak{g})$  be the BGG category associated to a triangular decomposition of a finite dimensional complex semisimple or reductive Lie algebra  $\mathfrak{g}$ . The BGG category  $\mathcal{O}(\mathfrak{g})$  and its projective functors play an important role in many algebraic categorifications, which can be seen from the following two facets. On one hand, the projective functors of  $\mathcal{O}(\mathfrak{g})$  are extensively used in categorifications of group algebras of finite Weyl groups and their Hecke algebras. In [10], Khovanov, *et al.* presented several examples about categorifications of various representations of the symmetric group  $S_n$  via projective functors acting on certain subcategories of the BGG category  $\mathcal{O}(\mathfrak{sl}_n)$ . Especially in [11] they categorified integral Specht modules over  $S_n$  and its Hecke algebra by some translation functors of  $\mathcal{O}(\mathfrak{sl}_n)$ . Mazorchuk and Stroppel [15] constructed a subcategory of  $\mathcal{O}(\mathfrak{g})$  on which the actions of translation functors categorify (right) cell modules and induced cell modules for Hecke algebras of finite Weyl groups. Basing on the results in [15], they [16] gave a categorification of Wedderburns basis for  $\mathbb{C}[S_n]$ . Moreover, Mazorchuk and Miemietz [14] reproved and extended the result in [15] by studying 2-representations of abstract 2-categories from a more systematic and more abstract prospective. In addition, Mazorchuk and Stroppel [17] applied graded versions of translation functors and a subcategory of the principal block of  $\mathcal{O}(\mathfrak{g})$  to categorifications of a parabolic Hecke module (see also [19]). On the other hand, the BGG category  $\mathcal{O}(\mathfrak{g})$  and its projective functors can be applied to categorifications of universal enveloping algebras of simple Lie algebras. In [1] Bernstein, *et al.* investigated a categorification of the  $n$ -th tensor power of the fundamental representation of  $U(\mathfrak{sl}_2)$  via certain singular blocks and projective functors of  $\mathcal{O}(\mathfrak{gl}_n)$  (see also [10]). Following [1] Sussan [20] generalized the case of  $\mathfrak{sl}_2$  in [1] to that of  $\mathfrak{sl}_k$  and studied  $\mathfrak{sl}_k$ -link invariants.

In [1] Bernstein, *et al.* raised a more difficult problem: categorifications of the representation theory of arbitrary simple Lie algebra  $\mathfrak{g}$ . The main purpose of this article is to study a categorification of the  $n$ -th tensor power of the spin representation of  $U(\mathfrak{so}(7, \mathbb{C}))$ . The main tools for our categorification are also the BGG category  $\mathcal{O}(\mathfrak{gl}_n)$  and its projective functors. Our work can be considered as a part of categorifications

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Email: yjxu2002@163.com, slyang@bjut.edu.cn

of the representation theory of  $U(\mathfrak{g})$  for the simple Lie algebra  $\mathfrak{g}$  of type  $B_3$ . In other words, we categorify the image of  $U(\mathfrak{so}(7, \mathbb{C}))$  under the algebra homomorphism  $\Phi : U(\mathfrak{so}(7, \mathbb{C})) \rightarrow \text{End}(V_{\text{sp}}^{\otimes n})$  corresponding to the  $n$ -th tensor power of the spin representation  $V_{\text{sp}}$  of  $U(\mathfrak{so}(7, \mathbb{C}))$ . In fact, as standard representations of the special orthogonal Lie algebras  $\mathfrak{so}(m, \mathbb{C})$ , the spin representations are especially important since they not only play a fundamental role in the realization of exceptional simple Lie algebras but also have many important applications in Lie group, geometry and physics (see [12] Chapters I.5 and IV.9, and [7] Chapter 20).

This paper is organized as follows. In Section 2, we collect the background material that will be necessary in the sequel. In Section 3, we obtain a categorification of the  $n$ -th tensor power of the spin representation of  $U(\mathfrak{so}(7, \mathbb{C}))$ . First, we categorify the underlying space of the  $n$ -th tensor power  $V_{\text{sp}}^{\otimes n}$  of the spin representation  $V_{\text{sp}}$  for  $U(\mathfrak{so}(7, \mathbb{C}))$  by using certain subcategories of  $\mathcal{O}(\mathfrak{gl}_n)$  (Theorem 3.1). Next we yield a categorification of the  $U(\mathfrak{so}(7, \mathbb{C}))$  action on  $V_{\text{sp}}^{\otimes n}$  by projective functors of  $\mathcal{O}(\mathfrak{gl}_n)$  (Theorem 3.3). Finally, we lift defining relations of  $U(\mathfrak{so}(7, \mathbb{C}))$  to natural isomorphisms between functors (Theorem 3.4).

Throughout, we denote by  $\mathbb{C}$ ,  $\mathbb{R}$  and  $\mathbb{Z}$  the complex number field, the real number field and the set of integers respectively.

## 2. PRELIMINARIES

We start by reviewing some basic results about the universal enveloping algebra of Lie algebra  $\mathfrak{so}(7, \mathbb{C})$  and the BGG category of a complex reductive Lie algebra.

As an associative algebra, the universal enveloping algebra  $U(\mathfrak{so}(7, \mathbb{C}))$  of the special orthogonal Lie algebra  $\mathfrak{so}(7, \mathbb{C})$  is generated by  $h_i, e_i, f_i (1 \leq i \leq 3)$  over  $\mathbb{C}$  which are subject to the following relations:

$$h_i h_j = h_j h_i, \quad e_i f_j - f_j e_i = \delta_{i,j} h_i,$$

$$h_i e_j - e_j h_i = a_{i,j} e_j, \quad h_i f_j - f_j h_i = -a_{i,j} f_j,$$

$$\sum_{k=0}^{1-a_{i,j}} (-1)^k \binom{1-a_{i,j}}{k} e_i^{1-a_{i,j}-k} e_j e_i^k = 0 \text{ for } i \neq j,$$

$$\sum_{k=0}^{1-a_{i,j}} (-1)^k \binom{1-a_{i,j}}{k} f_i^{1-a_{i,j}-k} f_j f_i^k = 0 \text{ for } i \neq j,$$

where  $a_{i,j} (1 \leq i, j \leq 3)$  are the entries of the Cartan matrix  $A = (a_{i,j})_{3 \times 3}$  of  $\mathfrak{so}(7, \mathbb{C})$ .

Let  $V_{\text{sp}} = \bigoplus_{i=0}^7 \mathbb{C} v_i$  be an 8-dimensional vector space over  $\mathbb{C}$ . Then  $V_{\text{sp}}$  is a  $U(\mathfrak{so}(7, \mathbb{C}))$ -module in the following way:

$$\begin{aligned} h_1 v_7 &= 0, h_1 v_6 = 0, h_1 v_5 = v_5, h_1 v_4 = v_4, h_1 v_3 = -v_3, h_1 v_2 = -v_2, h_1 v_1 = 0, h_1 v_0 = 0, \\ h_2 v_7 &= 0, h_2 v_6 = v_6, h_2 v_5 = -v_5, h_2 v_4 = 0, h_2 v_3 = 0, h_2 v_2 = v_2, h_2 v_1 = -v_1, h_2 v_0 = 0, \\ h_3 v_7 &= v_7, h_3 v_6 = -v_6, h_3 v_5 = v_5, h_3 v_4 = -v_4, h_3 v_3 = v_3, h_3 v_2 = -v_2, h_3 v_1 = v_1, h_3 v_0 = -v_0, \\ e_1 v_7 &= 0, e_1 v_6 = 0, e_1 v_5 = 0, e_1 v_4 = 0, e_1 v_3 = v_5, e_1 v_2 = v_4, e_1 v_1 = 0, e_1 v_0 = 0, \\ e_2 v_7 &= 0, e_2 v_6 = 0, e_2 v_5 = v_6, e_2 v_4 = 0, e_2 v_3 = 0, e_2 v_2 = 0, e_2 v_1 = v_2, e_2 v_0 = 0, \end{aligned}$$

$$\begin{aligned}
e_3 v_7 &= 0, e_3 v_6 = v_7, e_3 v_5 = 0, e_3 v_4 = v_5, e_3 v_3 = 0, e_3 v_2 = v_3, e_3 v_1 = 0, e_3 v_0 = v_1, \\
f_1 v_7 &= 0, f_1 v_6 = 0, f_1 v_5 = v_3, f_1 v_4 = v_2, f_1 v_3 = 0, f_1 v_2 = 0, f_1 v_1 = 0, f_1 v_0 = 0, \\
f_2 v_7 &= 0, f_2 v_6 = v_5, f_2 v_5 = 0, f_2 v_4 = 0, f_2 v_3 = 0, f_2 v_2 = v_1, f_2 v_1 = 0, f_2 v_0 = 0, \\
f_3 v_7 &= v_6, f_3 v_6 = 0, f_3 v_5 = v_4, f_3 v_4 = 0, f_3 v_3 = v_2, f_3 v_2 = 0, f_3 v_1 = v_0, f_3 v_0 = 0.
\end{aligned}$$

The  $U(\mathfrak{so}(7, \mathbb{C}))$ -module  $V_{\text{sp}}$  is called the spin representation of  $U(\mathfrak{so}(7, \mathbb{C}))$ .

For convenience, we fix some notations which we need in the sequel. All Lie algebras and their representations are defined over  $\mathbb{C}$ . Let  $\mathfrak{g}$  be a finite dimensional reductive Lie algebra with a fixed triangular decomposition  $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ . Denote by  $U(\mathfrak{g})$  the universal enveloping algebra of  $\mathfrak{g}$ ,  $Z(U(\mathfrak{g}))$  the center of  $U(\mathfrak{g})$  and  $\Theta$  the set of the central characters.  $W$  denotes the Weyl group of  $\mathfrak{g}$ .  $\rho$  is the half-sum of positive roots. Define the dot-action of  $W$  on  $\mathfrak{h}^*$  as follows:  $w \cdot \lambda = w(\lambda + \rho) - \rho$ . For  $\lambda \in \mathfrak{h}^*$ , let  $\theta_\lambda : Z(U(\mathfrak{g})) \rightarrow \mathbb{C}$  be the corresponding central character and  $M(\lambda)$  the Verma module with the highest weight  $\lambda$ .

Let  $\mathfrak{h}_{\text{dom}}^*$  be the set of all elements in  $\mathfrak{h}^*$  dominant with respect to the dot-action. Then there is a map  $\eta : \mathfrak{h}^* \rightarrow \mathfrak{h}_{\text{dom}}^*$  which maps  $\lambda$  to  $\theta_\lambda$  sets up a bijection between  $\mathfrak{h}_{\text{dom}}^*$  and  $\Theta$  (see [6], Section 7.4). The notation  $\mathcal{O}(\mathfrak{g})$  denotes the BGG category of  $\mathfrak{g}$  associated to the triangular decomposition  $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ . For any  $\theta \in \Theta$  denote by  $\mathcal{O}_\theta(\mathfrak{g})$  the full subcategory of  $\mathcal{O}(\mathfrak{g})$  whose objects are the modules  $M$  where

$$M = \left\{ m \in M \mid (z - \theta(z))^n \cdot m = 0 \text{ for some } n \in \mathbb{N} \text{ for each } z \in Z(U(\mathfrak{g})) \right\}.$$

The BGG category  $\mathcal{O}(\mathfrak{g})$  is the direct sum of the subcategories  $\mathcal{O}_\theta(\mathfrak{g})$  as  $\theta$  ranges over the central characters of the form  $\theta_\lambda$  (see [9], Section 1.12).

Now we give a brief introduction to projective functors.

Denote by  $\text{proj}_\theta$  the functor from  $\mathcal{O}(\mathfrak{g})$  to  $\mathcal{O}(\mathfrak{g})$  that, to a module  $M = \bigoplus_{\theta \in \Theta} M(\theta)$ , associates the  $\theta$ -component summand  $M(\theta)$  of  $M$ . Let  $F_V$  be the functor of tensoring with a finite-dimensional  $\mathfrak{g}$ -module  $V$ .

**Definition 2.1.**  $F : \mathcal{O}(\mathfrak{g}) \rightarrow \mathcal{O}(\mathfrak{g})$  is a projective functor if it is isomorphic to a direct summand of the functor  $F_V$  for some finite dimensional  $\mathfrak{g}$ -module  $V$ .

Denote by  $K(\mathcal{A})$  the Grothendieck group of an abelian or triangulated category  $\mathcal{A}$ . It is the free abelian group generated by the symbols  $[M]$  where  $M$  is an object of  $\mathcal{A}$ . The only relations in this group are of the form  $[N] = [M] + [P]$  when there is a short exact sequence or distinguished triangle of the form  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ . The image of an object  $M$  and an exact functor  $F$  in the Grothendieck group will be denoted by  $[M]$  and  $[F]$  respectively.

The following properties of projective functors can be found in Section 3.2 or 3.4 of [2].

- Proposition 2.2.** (1) *Projective functors are exact.*  
(2) *Any direct sum of projective functors is a projective functor.*  
(3) *Any composition of projective functors is a projective functor.*  
(4) *The functor  $\text{proj}_\theta : \mathcal{O}(\mathfrak{g}) \rightarrow \mathcal{O}(\mathfrak{g})$  is a projective functor.*  
(5) *Let  $F, G$  be projective functors. If  $[F] = [G]$ , then  $F \cong G$ .*

Fix a central character  $\theta$ , then

$$(2.1) \quad \left\{ [M(\lambda)] \mid \theta = \theta_\lambda \right\} = \left\{ [M(\mu)] \mid \mu \in W \cdot \lambda \right\}$$

forms a  $\mathbb{Z}$ -basis of the Grothendieck group  $K(\mathcal{O}_\theta(\mathfrak{g}))$  (see [9], Section 1.10 and 1.12). The following proposition shows that this basis is handy for writing the action of projective functors on the Grothendieck group of  $\mathcal{O}(\mathfrak{g})$  (see [1], Section 2.3.2, and [2], Section 1.12).

**Proposition 2.3.** *Let  $V$  be a finite-dimensional  $\mathfrak{g}$ -module,  $\mu_1, \dots, \mu_m$  the multiset of weights of  $V$ , i.e., there is a basis  $v_1, v_2, \dots, v_m$  of  $V$  such that the weight of the vector  $v_i$  equals  $\mu_i$ ,  $M(\lambda)$  the Verma module with the highest weight  $\lambda$ , then we have  $[V \otimes M(\lambda)] = \sum_{i=1}^m [M(\lambda + \mu_i)]$  in the Grothendieck group  $K(\mathcal{O}(\mathfrak{g}))$ .*

For unexplained concepts and notations, we refer the reader to [2, 3, 7, 8, 9, 13].

### 3. CATEGORIFICATION OF THE SPIN REPRESENTATION OF $U(\mathfrak{so}(7, \mathbb{C}))$

This section is to obtain a categorification of the  $n$ -th tensor power  $V_{\text{sp}}^{\otimes n}$  of the spin representation  $V_{\text{sp}}$  for  $U(\mathfrak{so}(7, \mathbb{C}))$  in the following three steps.

- (1) Categorifying the underlying space of the  $n$ -th tensor power  $V_{\text{sp}}^{\otimes n}$  of the spin representation  $V_{\text{sp}}$  of  $U(\mathfrak{so}(7, \mathbb{C}))$  by using certain subcategories of the BGG category  $\mathcal{O}(\mathfrak{gl}_n)$ .
- (2) Yielding a categorification of the  $U(\mathfrak{so}(7, \mathbb{C}))$  action on  $V_{\text{sp}}^{\otimes n}$  by projective functors of  $\mathcal{O}(\mathfrak{gl}_n)$ .
- (3) Lifting defining relations of  $U(\mathfrak{so}(7, \mathbb{C}))$  to natural isomorphisms between functors.

We fix once and for all a triangular decomposition  $\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$  of the Lie algebra  $\mathfrak{gl}_n$ . The Weyl group of  $\mathfrak{gl}_n$  is isomorphic to the symmetric group  $S_n$ . Choose a standard orthogonal basis  $\varepsilon_1, \dots, \varepsilon_n$  in the Euclidean space  $\mathbb{R}^n$  and identify the complexification  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n$  with the dual  $\mathfrak{h}^*$  of Cartan subalgebra so that  $R_+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \}$  is the set of positive roots and  $\beta_i = \varepsilon_i - \varepsilon_{i+1}$ ,  $1 \leq i \leq n-1$  are simple roots. The generator  $s_i$  of the Weyl group  $W = S_n$  acts on  $\mathfrak{h}^*$  by permuting  $\varepsilon_i$  and  $\varepsilon_{i+1}$ . Denote by  $\rho$  the half-sum of positive roots

$$\rho = \frac{n-1}{2} \varepsilon_1 + \frac{n-3}{2} \varepsilon_2 + \dots + \frac{1-n}{2} \varepsilon_n.$$

We denote by  $[0, 7]$  the integers  $0 \leq k \leq 7$ . For a sequence  $(a_1, \dots, a_n) \in [0, 7]^n$  we denote by  $M(a_1, \dots, a_n)$  the Verma module with the highest weight  $a_1 \varepsilon_1 + \dots + a_n \varepsilon_n - \rho$ .

Let  $\mathbf{D}$  be the set of all 8-tuples of nonnegative integers  $\mathbf{d} = (d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7)$  such that  $\sum_{k=0}^7 d_k = n$ . We define the following equivalence relation  $\sim$  on  $\mathbf{D}$ :

$$\mathbf{d} \sim \mathbf{d}' \Leftrightarrow \begin{cases} d_7 + d_6 + d_5 + d_4 - d_3 - d_2 - d_1 - d_0 = d'_7 + d'_6 + d'_5 + d'_4 - d'_3 - d'_2 - d'_1 - d'_0, \\ d_7 + d_6 - d_5 - d_4 + d_3 + d_2 - d_1 - d_0 = d'_7 + d'_6 - d'_5 - d'_4 + d'_3 + d'_2 - d'_1 - d'_0, \\ d_7 - d_6 + d_5 - d_4 + d_3 - d_2 + d_1 - d_0 = d'_7 - d'_6 + d'_5 - d'_4 + d'_3 - d'_2 + d'_1 - d'_0, \end{cases}$$

for any  $\mathbf{d} = (d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7)$ ,  $\mathbf{d}' = (d'_0, d'_1, d'_2, d'_3, d'_4, d'_5, d'_6, d'_7) \in \mathbf{D}$ . In the following  $[\mathbf{d}]$  and  $\widetilde{\mathbf{D}}$  represent the equivalent class of  $\mathbf{d}$  and the set of all the equivalent classes respectively.

The spin representation  $V_{\text{sp}}$  has the weight space decomposition  $V_{\text{sp}} = \bigoplus_{k=0}^7 V_k$ , where  $V_k = \mathbb{C}v_k$  for  $0 \leq k \leq 7$  (see [8] Chapter 2). For  $\mathbf{d}' \in \mathbf{D}$  and  $(a_1, \dots, a_n) \in [0, 7]^n$ , we define the following condition:

$$(3.1) \quad \# \{ a_m \mid a_m = k, 1 \leq m \leq n \} = d'_k \text{ for } 0 \leq k \leq 7.$$

It follows that  $V_{\text{sp}}^{\otimes n}$  has the weight space decomposition  $V_{\text{sp}}^{\otimes n} = \bigoplus_{[\mathbf{d}] \in \widetilde{\mathbf{D}}} (V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$ , where  $(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  is the  $\mathbb{C}$ -linear space spanned by

$$B'_{[\mathbf{d}]} := \left\{ v_{a_1} \otimes v_{a_2} \otimes \dots \otimes v_{a_n} \mid (a_1, \dots, a_n) \in [0, 7]^n \text{ satisfying the condition (3.1) for some } \mathbf{d}' \in [\mathbf{d}] \right\}.$$

From now on, we denote by  ${}^{\mathbb{Z}}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  the  $\mathbb{Z}$ -module spanned by  $B'_{[\mathbf{d}]}$  and  ${}^{\mathbb{Z}}V_{\text{sp}}^{\otimes n} := \bigoplus_{[\mathbf{d}] \in \widetilde{\mathbf{D}}} {}^{\mathbb{Z}}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$ . It is easy to see that  $\mathbb{C} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]} = (V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  and  $\mathbb{C} \otimes_{\mathbb{Z}} {}^{\mathbb{Z}}V_{\text{sp}}^{\otimes n} = V_{\text{sp}}^{\otimes n}$ . For each  $\mathbf{d} \in \mathbf{D}$ , set  $\lambda_{\mathbf{d}} = \sum_{i=0}^7 \sum_{j=1}^{d_i} (7-i) \varepsilon_{d_0+\dots+d_{i-1}+j}$ . Denote by  $\theta_{\mathbf{d}} = \eta(\lambda_{\mathbf{d}} - \rho)$  the corresponding central character of  $\mathfrak{gl}_n$  under the map  $\eta : \mathfrak{h}^* \rightarrow \Theta$ . We define  $\mathcal{O}_{\mathbf{d}} := \mathcal{O}_{\theta_{\mathbf{d}}}(\mathfrak{gl}_n)$ ,  $\mathcal{O}_{[\mathbf{d}]} := \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} \mathcal{O}_{\mathbf{d}'}$  and  $\mathcal{O}^n := \bigoplus_{[\mathbf{d}] \in \widetilde{\mathbf{D}}} \mathcal{O}_{[\mathbf{d}]}$ .

Now we are prepared to realize  ${}^{\mathbb{Z}}V_{\text{sp}}^{\otimes n}$  and its weight space  ${}^{\mathbb{Z}}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  for any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$  as the Grothendieck groups of the categories  $\mathcal{O}^n$  and  $\mathcal{O}_{[\mathbf{d}]}$  respectively. Indeed, we have the following result.

**Theorem 3.1.** *There exists an isomorphism of abelian groups  $\gamma_n : K(\mathcal{O}^n) \rightarrow {}^{\mathbb{Z}}V_{\text{sp}}^{\otimes n}$  given by*

$$\gamma_n([M(a_1, \dots, a_n)]) = v_{a_1} \otimes v_{a_2} \otimes \dots \otimes v_{a_n}$$

for any sequence  $(a_1, \dots, a_n) \in [0, 7]^n$ . Moreover, the restriction of  $\gamma_n$  on  $K(\mathcal{O}_{[\mathbf{d}]})$  is an abelian group isomorphism between  $K(\mathcal{O}_{[\mathbf{d}]})$  and  ${}^{\mathbb{Z}}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  for any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ .

*Proof.* To prove the theorem, it is sufficient to prove that  $\gamma_n : K(\mathcal{O}_{[\mathbf{d}]}) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow {}^{\mathbb{Z}}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  is an abelian group isomorphism for any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ . Indeed, the above abelian group isomorphism will be obvious if we note the following facts.

For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$  and  $\mathbf{d}' \in [\mathbf{d}]$  it is seen from (2.1) that the set of all the symbols  $[M(a_1, \dots, a_n)]$  satisfying the condition (3.1) is a  $\mathbb{Z}$ -basis of the Grothendieck group  $K(\mathcal{O}_{\mathbf{d}'})$ . We denote this  $\mathbb{Z}$ -basis by  $B_{\mathbf{d}'}$ . It follows that  $B_{[\mathbf{d}]} = \bigcup_{\mathbf{d}' \in [\mathbf{d}]} B_{\mathbf{d}'}$  is a  $\mathbb{Z}$ -basis of the Grothendieck group  $K(\mathcal{O}_{[\mathbf{d}]})$ . On the other hand, if we denote by  $B'_{\mathbf{d}'}$  the set of  $v_{a_1} \otimes v_{a_2} \otimes \dots \otimes v_{a_n}$  such that the sequence  $(a_1, \dots, a_n) \in [0, 7]^n$  satisfies (3.1), then  $B'_{[\mathbf{d}]} := \bigcup_{\mathbf{d}' \in [\mathbf{d}]} B'_{\mathbf{d}'}$  is a  $\mathbb{Z}$ -basis of the weight space  ${}^{\mathbb{Z}}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$ .  $\square$

Let  $L_n$  be the  $n$ -dimensional fundamental representation of  $\mathfrak{gl}_n$  with weights  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$  and the corresponding weight vectors  $u_1, u_2, \dots, u_n$ . Then its dual representation  $L_n^*$  has weights  $-\varepsilon_1, -\varepsilon_2, \dots, -\varepsilon_n$ . In addition, we recall some submodules of  $L_n^{\otimes 2}$  and  $(L_n^*)^{\otimes 2}$  which will be used to construct the functors for our categorification. Denote by  $\text{Sym}^2(L_n)$  the symmetric square of  $L_n$ , i.e., the submodule of  $L_n^{\otimes 2}$  spanned by  $u_i \otimes u_i (1 \leq i \leq n)$  and  $u_i \otimes u_j + u_j \otimes u_i (1 \leq i < j \leq n)$ , and denote by  $\text{Alt}^2(L_n)$  the alternative square of  $L_n$ , i.e., the submodule of  $L_n^{\otimes 2}$  spanned by  $u_i \otimes u_j - u_j \otimes u_i (1 \leq i < j \leq n)$ . Similarly, we denote by  $\text{Sym}^2(L_n^*)$  and  $\text{Alt}^2(L_n^*)$  the symmetric square and the alternative square of  $L_n^*$  respectively. In the following, we define  $\mathcal{O}_{\mathbf{d}}$  to be the trivial subcategory of  $\mathcal{O}(\mathfrak{gl}_n)$  for  $\mathbf{d} \notin \mathbf{D}$ . For  $\mathbf{d} \in \mathbf{D}$  let  $\mathbf{d}_i$  denote the fact that one subtracts 1 from the coefficient at place  $i$ , and  $\mathbf{d}^i$  the fact that one adds 1 to the coefficient at place  $i$ . Then  $\mathbf{d}_i^j$  means that one subtracts 1 from the coefficient at place  $i$  and adds 1 to the coefficient at place  $j$ .

To categorify the action of  $U(\mathfrak{so}(7, \mathbb{C}))$  on  $V_{\text{sp}}^{\otimes n}$ , we introduce a series of projective functors of  $\mathcal{O}(\mathfrak{gl}_n)$ .

For  $\mathbf{d} = (d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7) \in \mathbf{D}$ , define

$$c_1(\mathbf{d}) := d_5 + d_4 - d_3 - d_2,$$

$$c_2(\mathbf{d}) := d_6 - d_5 + d_2 - d_1,$$

$$c_3(\mathbf{d}) := d_7 - d_6 + d_5 - d_4 + d_3 - d_2 + d_1 - d_0,$$

and for  $1 \leq i \leq 3$ , denote by  $\text{sgn}(c_i(\mathbf{d}))$  the sign function of  $c_i(\mathbf{d})$ , i.e.,

$$\text{sgn}(c_i(\mathbf{d})) = \begin{cases} 1, & \text{if } c_i(\mathbf{d}) > 0, \\ 0, & \text{if } c_i(\mathbf{d}) = 0, \\ -1, & \text{if } c_i(\mathbf{d}) < 0. \end{cases}$$

Then set

$$\mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) = (\text{Id}_{\mathcal{O}_{[\mathbf{d}]}})^{\oplus \text{sgn}(c_i(\mathbf{d}))c_i(\mathbf{d})} : \mathcal{O}_{[\mathbf{d}]} \rightarrow \mathcal{O}_{[\mathbf{d}]},$$

where  $\text{Id}_{\mathcal{O}_{[\mathbf{d}]}}$  is the identity functor of  $\mathcal{O}_{[\mathbf{d}]}$ . From the definition of the equivalence relation  $\sim$  on  $\mathbf{D}$  we can see that the functors  $\mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) (1 \leq i \leq 3)$  are independent on the choice of the representative  $\mathbf{d}$  of  $[\mathbf{d}]$ .

For  $\mathbf{d} \in \mathbf{D}$  denote

$$\begin{aligned} \mathcal{E}_1^{+2}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_2^4}} \circ F_{\text{Sym}^2(L_n)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_2^4}, \\ \mathcal{E}_1^{-2}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_2^4}} \circ F_{\text{Alt}^2(L_n)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_2^4}, \\ \mathcal{E}_1^{+3}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_3^5}} \circ F_{\text{Sym}^2(L_n)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_3^5}, \\ \mathcal{E}_1^{-3}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_3^5}} \circ F_{\text{Alt}^2(L_n)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_3^5}. \end{aligned}$$

For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , we set

$$\begin{aligned} \mathcal{E}_1^+([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{E}_1^{+2}(\mathbf{d}') \oplus \mathcal{E}_1^{+3}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \rightarrow \mathcal{O}_{[\mathbf{d}_1]^{-}}, \\ \mathcal{E}_1^-([\mathbf{d}]) &= \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{E}_1^{-2}(\mathbf{d}') \oplus \mathcal{E}_1^{-3}(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \rightarrow \mathcal{O}_{[\mathbf{d}_1]^{-}}, \end{aligned}$$

where  $[\mathbf{d}_1]^{-} = [\mathbf{d}_3^5] = [\mathbf{d}_2^4]$ .

For  $\mathbf{d} \in \mathbf{D}$  denote

$$\begin{aligned} \mathcal{E}_2^1(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_1^2}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_1^2}, \\ \mathcal{E}_2^5(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_5^6}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_5^6}. \end{aligned}$$

For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , set

$$\mathcal{E}_2([\mathbf{d}]) = \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{E}_2^1(\mathbf{d}') \oplus \mathcal{E}_2^5(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \rightarrow \mathcal{O}_{[\mathbf{d}_2]^{-}},$$

where  $[\mathbf{d}_2]^{-} = [\mathbf{d}_1^2] = [\mathbf{d}_5^6]$ .

For  $\mathbf{d} \in \mathbf{D}$  denote

$$\begin{aligned} \mathcal{E}_3^0(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_0^1}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_0^1}, \\ \mathcal{E}_3^2(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_2^3}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_2^3}, \\ \mathcal{E}_3^4(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_4^5}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_4^5}, \\ \mathcal{E}_3^6(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_6^7}} \circ F_{L_n} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_6^7}. \end{aligned}$$

For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , set

$$\mathcal{E}_3([\mathbf{d}]) = \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{E}_3^0(\mathbf{d}') \oplus \mathcal{E}_3^2(\mathbf{d}') \oplus \mathcal{E}_3^4(\mathbf{d}') \oplus \mathcal{E}_3^6(\mathbf{d}')) : \mathcal{O}_{[\mathbf{d}]} \rightarrow \mathcal{O}_{[\mathbf{d}_3]^{-}},$$

where  $[\mathbf{d}_3]^{-} = [\mathbf{d}_0^1] = [\mathbf{d}_2^3] = [\mathbf{d}_4^5] = [\mathbf{d}_6^7]$ .

For  $\mathbf{d} \in \mathbf{D}$  denote

$$\begin{aligned} \mathcal{F}_1^{+4}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_4^2}} \circ F_{\text{Sym}^2(L_n^*)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_4^2}, \\ \mathcal{F}_1^{-4}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_4^2}} \circ F_{\text{Alt}^2(L_n^*)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_4^2}, \\ \mathcal{F}_1^{+5}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_5^3}} \circ F_{\text{Sym}^2(L_n^*)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_5^3}, \\ \mathcal{F}_1^{-5}(\mathbf{d}) &:= \text{proj}_{\theta_{\mathbf{d}_5^3}} \circ F_{\text{Alt}^2(L_n^*)} : \mathcal{O}_{\mathbf{d}} \rightarrow \mathcal{O}_{\mathbf{d}_5^3} \end{aligned}$$

For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , set

$$\mathcal{F}_1^+([\mathbf{d}]) = \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{F}_1^{+4}(\mathbf{d}') \oplus \mathcal{F}_1^{+5}(\mathbf{d}')) : \emptyset_{[\mathbf{d}]} \rightarrow \emptyset_{[\vec{\mathbf{d}}_1]},$$

$$\mathcal{F}_1^-([\mathbf{d}]) = \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{F}_1^{-4}(\mathbf{d}') \oplus \mathcal{F}_1^{-5}(\mathbf{d}')) : \emptyset_{[\mathbf{d}]} \rightarrow \emptyset_{[\vec{\mathbf{d}}_1]},$$

where  $[\vec{\mathbf{d}}_1] = [\mathbf{d}_4^2] = [\mathbf{d}_5^3]$ .

For  $\mathbf{d} \in \mathbf{D}$  denote

$$\mathcal{F}_2^2(\mathbf{d}) := \text{proj}_{\theta_{\mathbf{d}_1^2}} \circ F_{L_n^*} : \emptyset_{\mathbf{d}} \rightarrow \emptyset_{\mathbf{d}_1^2},$$

$$\mathcal{F}_2^6(\mathbf{d}) := \text{proj}_{\theta_{\mathbf{d}_6^5}} \circ F_{L_n^*} : \emptyset_{\mathbf{d}} \rightarrow \emptyset_{\mathbf{d}_6^5}.$$

For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , set

$$\mathcal{F}_2([\mathbf{d}]) = \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{F}_2^2(\mathbf{d}') \oplus \mathcal{F}_2^6(\mathbf{d}')) : \emptyset_{[\mathbf{d}]} \rightarrow \emptyset_{[\vec{\mathbf{d}}_2]},$$

where  $[\vec{\mathbf{d}}_2] = [\mathbf{d}_2^1] = [\mathbf{d}_6^5]$ .

For  $\mathbf{d} \in \mathbf{D}$  denote

$$\mathcal{F}_3^1(\mathbf{d}) := \text{proj}_{\theta_{\mathbf{d}_1^0}} \circ F_{L_n^*} : \emptyset_{\mathbf{d}} \rightarrow \emptyset_{\mathbf{d}_1^0},$$

$$\mathcal{F}_3^3(\mathbf{d}) := \text{proj}_{\theta_{\mathbf{d}_3^2}} \circ F_{L_n^*} : \emptyset_{\mathbf{d}} \rightarrow \emptyset_{\mathbf{d}_3^2},$$

$$\mathcal{F}_3^5(\mathbf{d}) := \text{proj}_{\theta_{\mathbf{d}_5^4}} \circ F_{L_n^*} : \emptyset_{\mathbf{d}} \rightarrow \emptyset_{\mathbf{d}_5^4},$$

$$\mathcal{F}_3^7(\mathbf{d}) := \text{proj}_{\theta_{\mathbf{d}_7^6}} \circ F_{L_n^*} : \emptyset_{\mathbf{d}} \rightarrow \emptyset_{\mathbf{d}_7^6}.$$

For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , set

$$\mathcal{F}_3([\mathbf{d}]) = \bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{F}_3^1(\mathbf{d}') \oplus \mathcal{F}_3^3(\mathbf{d}') \oplus \mathcal{F}_3^5(\mathbf{d}') \oplus \mathcal{F}_3^7(\mathbf{d}')) : \emptyset_{[\mathbf{d}]} \rightarrow \emptyset_{[\vec{\mathbf{d}}_3]},$$

where  $[\vec{\mathbf{d}}_3] = [\mathbf{d}_1^0] = [\mathbf{d}_3^2] = [\mathbf{d}_5^4] = [\mathbf{d}_7^6]$ .

It can be seen from Proposition 2.2 (1), (2), (3) and (4) that the above functors we introduce are exact and projective functors. Therefore, they can induce abelian group homomorphisms of the corresponding Grothendieck groups. By Proposition 2.3 and direct calculations we can obtain the following explicit formulas of their induced homomorphisms on the basis element  $[M(a_1, \dots, a_n)] \in B_{\mathbf{d}}$ , which will be used in checking the commutativity of the diagrams in Proposition 3.2.

$$\begin{aligned} (3.2) \quad & [\mathcal{E}_1^{+2}(\mathbf{d})]([M(a_1, \dots, a_n)]) \\ &= \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m + 2, a_{m+1}, \dots, a_n)] \\ &+ \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (2, 5) \text{ or } (3, 2)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)]. \end{aligned}$$

$$\begin{aligned} (3.3) \quad & [\mathcal{E}_1^{-2}(\mathbf{d})]([M(a_1, \dots, a_n)]) \\ &= \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (2, 3) \text{ or } (3, 2)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)]. \end{aligned}$$

$$\begin{aligned}
(3.4) \quad & [\mathcal{E}_1^{+3}(\mathbf{d})](M(a_1, \dots, a_n)) \\
&= \sum_{\substack{m=1, \\ a_m=3}}^n [M(a_1, \dots, a_{m-1}, a_m + 2, a_{m+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (3,4) \text{ or } (4,3)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$

$$\begin{aligned}
(3.5) \quad & [\mathcal{E}_1^{-3}(\mathbf{d})](M(a_1, \dots, a_n)) \\
&= \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (3,4) \text{ or } (4,3)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$

$$(3.6) \quad [\mathcal{E}_2^1(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=1}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)].$$

$$(3.7) \quad [\mathcal{E}_2^5(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=5}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)].$$

$$(3.8) \quad [\mathcal{E}_3^0(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=0}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)].$$

$$(3.9) \quad [\mathcal{E}_3^2(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)].$$

$$(3.10) \quad [\mathcal{E}_3^4(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=4}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)].$$

$$(3.11) \quad [\mathcal{E}_3^6(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=6}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)].$$

$$\begin{aligned}
(3.12) \quad & [\mathcal{F}_1^{+4}(\mathbf{d})](M(a_1, \dots, a_n)) \\
&= \sum_{\substack{m=1, \\ a_m=4}}^n [M(a_1, \dots, a_{m-1}, a_m - 2, a_{m+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (3,4) \text{ or } (4,3)}} [M(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$

$$\begin{aligned}
(3.13) \quad & [\mathcal{F}_1^{-4}(\mathbf{d})](M(a_1, \dots, a_n)) \\
&= \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (3,4) \text{ or } (4,3)}} [M(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$



$$\begin{aligned}
(3.14) \quad & [\mathcal{F}_1^{+5}(\mathbf{d})](M(a_1, \dots, a_n)) \\
&= \sum_{\substack{m=1, \\ a_m=5}}^n [M(a_1, \dots, a_{m-1}, a_m - 2, a_{m+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (4,5) \text{ or } (5,5)}} [M(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$

$$\begin{aligned}
(3.15) \quad & [\mathcal{F}_1^{-5}(\mathbf{d})](M(a_1, \dots, a_n)) \\
&= \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (4,5) \text{ or } (5,4)}} [M(a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$

$$(3.16) \quad [\mathcal{F}_2^2(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n)].$$

$$(3.17) \quad [\mathcal{F}_2^6(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=6}}^n [M(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n)].$$

$$(3.18) \quad [\mathcal{F}_3^1(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=1}}^n [M(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n)].$$

$$(3.19) \quad [\mathcal{F}_3^3(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=3}}^n [M(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n)].$$

$$(3.20) \quad [\mathcal{F}_3^5(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=5}}^n [M(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n)].$$

$$(3.21) \quad [\mathcal{F}_3^7(\mathbf{d})](M(a_1, \dots, a_n)) = \sum_{\substack{m=1, \\ a_m=7}}^n [M(a_1, \dots, a_{m-1}, a_m - 1, a_{m+1}, \dots, a_n)].$$

Indeed, we check the formula (3.2) as follows:

$$\begin{aligned}
& [\mathcal{E}_1^{+2}(\mathbf{d})](M(a_1, \dots, a_n)) \\
&= [\text{proj}_{\theta_{d_1^2}}(\text{Sym}^2(L_n) \otimes M(a_1, \dots, a_n))] \\
&= [\text{proj}_{\theta_{d_1^2}}](\sum_{1 \leq i \leq j \leq n} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)]) \\
&= \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m + 2, a_{m+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (2,3) \text{ or } (3,2)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$

Note that the second equality is obtained by Proposition 2.3, while others are obvious by the definitions of the projective functors  $\mathcal{E}_1^{+2}(\mathbf{d})$ ,  $F_{\text{Sym}^2(L_n)}$  and  $\text{proj}_{\theta}$ .

Similarly, we can get the formulas from (3.3) to (3.21).

Now, by the formulas (3.2)–(3.21), a categorification of the action of  $U(\mathfrak{so}(7, \mathbb{C}))$  on the  $n$ -th tensor power of its spin representation can be obtained as follows.

**Proposition 3.2.** (1) For any  $1 \leq i \leq 3$  and  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , the action of  $h_i$  on  $(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  can be categorified by the exact functor  $\mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])$ , which means that the following diagram commutes:

$$\begin{array}{ccc} K(\emptyset_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]} \\ [\mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])] \downarrow & & \downarrow \text{sgn}(c_i(\mathbf{d}))h_i \\ K(\emptyset_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]} \end{array}$$

(2) (a) For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , the restriction of  $e_1$  from  $(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  to  $(V_{\text{sp}}^{\otimes n})_{[\tilde{\mathbf{d}}_1]}$  can be categorified by a pair of exact functors  $(\mathcal{E}_1^+([\mathbf{d}]), \mathcal{E}_1^-([\mathbf{d}]))$ , which means that the following diagram commutes:

$$\begin{array}{ccc} K(\emptyset_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]} \\ [\mathcal{E}_1^+([\mathbf{d}])] - [\mathcal{E}_1^-([\mathbf{d}])] \downarrow & & \downarrow e_1 \\ K(\emptyset_{[\tilde{\mathbf{d}}_1]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\tilde{\mathbf{d}}_1]} \end{array}$$

(b) For any  $2 \leq i \leq 3$  and  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , the restriction of  $e_i$  from  $(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  to  $(V_{\text{sp}}^{\otimes n})_{[\tilde{\mathbf{d}}_i]}$  can be categorified by the exact functor  $\mathcal{E}_i([\mathbf{d}])$ , which means that the following diagram commutes:

$$\begin{array}{ccc} K(\emptyset_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]} \\ [\mathcal{E}_i([\mathbf{d}])] \downarrow & & \downarrow e_i \\ K(\emptyset_{[\tilde{\mathbf{d}}_i]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\tilde{\mathbf{d}}_i]} \end{array}$$

(3) (a) For any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , the restriction of  $f_1$  from  $(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  to  $(V_{\text{sp}}^{\otimes n})_{[\vec{\mathbf{d}}_1]}$  can be categorified by a pair of exact functors  $(\mathcal{F}_1^+([\mathbf{d}]), \mathcal{F}_1^-([\mathbf{d}]))$ , which means that the following diagram commutes:

$$\begin{array}{ccc} K(\emptyset_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]} \\ [\mathcal{F}_1^+([\mathbf{d}])] - [\mathcal{F}_1^-([\mathbf{d}])] \downarrow & & \downarrow f_1 \\ K(\emptyset_{[\vec{\mathbf{d}}_1]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\vec{\mathbf{d}}_1]} \end{array}$$

(b) For any  $2 \leq i \leq 3$  and  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$ , the restriction of  $f_i$  from  $(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]}$  to  $(V_{\text{sp}}^{\otimes n})_{[\vec{\mathbf{d}}_i]}$  can be categorified by the exact functor  $\mathcal{F}_i([\mathbf{d}])$ , which means that the following diagram commutes:

$$\begin{array}{ccc} K(\emptyset_{[\mathbf{d}]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\mathbf{d}]} \\ [\mathcal{F}_i([\mathbf{d}])] \downarrow & & \downarrow f_i \\ K(\emptyset_{[\vec{\mathbf{d}}_i]}) & \xrightarrow{\gamma_n} & \mathbb{Z}(V_{\text{sp}}^{\otimes n})_{[\vec{\mathbf{d}}_i]} \end{array}$$

*Proof.* Here we check (1) and (2) in some cases. Other cases can be verified similarly.

(1) To check  $\gamma_n \circ [\mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])] = \text{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n$  is equivalent to check

$$\gamma_n \circ [\mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])](M(a_1, \dots, a_n)) = \text{sgn}(c_i(\mathbf{d}))h_i \circ \gamma_n(M(a_1, \dots, a_n))$$

for any  $[M(a_1, \dots, a_n)] \in B_{[\mathbf{d}]} = \bigcup_{\mathbf{d}' \in [\mathbf{d}]} B_{\mathbf{d}'}$ . In fact, we have

$$\begin{aligned}
& \gamma_n \circ [\mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}])](M(a_1, \dots, a_n)) \\
&= |c_i(\mathbf{d})| \gamma_n([M(a_1, \dots, a_n)]) \\
&= |c_i(\mathbf{d})| (v_{a_1} \otimes v_{a_2} \otimes \dots \otimes v_{a_n}) \\
&= \text{sgn}(c_i(\mathbf{d})) c_i(\mathbf{d}) (v_{a_1} \otimes v_{a_2} \otimes \dots \otimes v_{a_n}) \\
&= \text{sgn}(c_i(\mathbf{d})) h_i \circ \gamma_n([M(a_1, \dots, a_n)]).
\end{aligned}$$

(2) (a) To verify the commutativity of the diagram in (a), it suffices to check  $\gamma_n \circ ([\mathcal{E}_1^+([\mathbf{d}])] - [\mathcal{E}_1^-([\mathbf{d}])]) = e_1 \circ \gamma_n$ . Indeed, for any  $[\mathbf{d}] \in \tilde{\mathbf{D}}$  and  $[M(a_1, \dots, a_n)] \in B_{[\mathbf{d}]} = \bigcup_{\mathbf{d}' \in [\mathbf{d}]} B_{\mathbf{d}'}$ , we assume  $[M(a_1, \dots, a_n)] \in B_{\mathbf{d}_0}$  for some  $\mathbf{d}_0 \in [\mathbf{d}]$ , by (3.2)–(3.5), we have the following formulas:

$$\begin{aligned}
& [\mathcal{E}_1^+([\mathbf{d}])](M(a_1, \dots, a_n)) \\
&= [\bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{E}_1^{+2}(\mathbf{d}') \oplus \mathcal{E}_1^{+3}(\mathbf{d}')) M(a_1, \dots, a_n)] \\
&= [(\mathcal{E}_1^{+2}(\mathbf{d}_0) \oplus \mathcal{E}_1^{+3}(\mathbf{d}_0)) M(a_1, \dots, a_n)] \\
&= \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m + 2, a_{m+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{m=1, \\ a_m=3}}^n [M(a_1, \dots, a_{m-1}, a_m + 2, a_{m+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (2,3) \text{ or } (3,2)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (3,4) \text{ or } (4,3)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)],
\end{aligned}$$

and

$$\begin{aligned}
& [\mathcal{E}_1^-([\mathbf{d}])](M(a_1, \dots, a_n)) \\
&= [\bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{E}_1^{-2}(\mathbf{d}') \oplus \mathcal{E}_1^{-3}(\mathbf{d}')) M(a_1, \dots, a_n)] \\
&= [(\mathcal{E}_1^{-2}(\mathbf{d}_0) \oplus \mathcal{E}_1^{-3}(\mathbf{d}_0)) M(a_1, \dots, a_n)] \\
&= \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (2,3) \text{ or } (3,2)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{1 \leq i < j \leq n, \\ (a_i, a_j) = (3,4) \text{ or } (4,3)}} [M(a_1, \dots, a_{i-1}, a_i + 1, a_{i+1}, \dots, a_{j-1}, a_j + 1, a_{j+1}, \dots, a_n)].
\end{aligned}$$

It follows that

$$\begin{aligned}
& \gamma_n \circ ([\mathcal{E}_1^+([\mathbf{d}])] - [\mathcal{E}_1^-([\mathbf{d}])])(M(a_1, \dots, a_n)) \\
&= \gamma_n \left( \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m + 2, a_{m+1}, \dots, a_n)] \right) + \gamma_n \left( \sum_{\substack{m=1, \\ a_m=3}}^n [M(a_1, \dots, a_{m-1}, a_m + 2, a_{m+1}, \dots, a_n)] \right) \\
&= \sum_{\substack{m=1, \\ a_m=2}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) + \sum_{\substack{m=1, \\ a_m=3}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
& e_1 \circ \gamma_n([M(a_1, \dots, a_n)]) \\
&= e_1(v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) \\
&= \sum_{m=1}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes e_1 v_{a_m} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) \\
&= \sum_{\substack{m=1, \\ a_m=2}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) \\
&\quad + \sum_{\substack{m=1, \\ a_m=3}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}).
\end{aligned}$$

Therefore  $\gamma_n \circ ([\mathcal{E}_1^+(\mathbf{d})] - [\mathcal{E}_1^-(\mathbf{d})]) = e_1 \circ \gamma_n$ .

(b) We give the proof of the case  $i = 3$ .

In the following we will prove  $\gamma_n \circ [\mathcal{E}_3(\mathbf{d})] = e_3 \circ \gamma_n$  which means the diagram in this case commutes. In fact, for any  $[\mathbf{d}] \in \widetilde{\mathbf{D}}$  and  $[M(a_1, \dots, a_n)] \in B_{[\mathbf{d}]} = \bigcup_{\mathbf{d}' \in [\mathbf{d}]} B_{\mathbf{d}'}$ , we only need to check

$$\gamma_n \circ [\mathcal{E}_3([\mathbf{d}])](M(a_1, \dots, a_n)) = e_3 \circ \gamma_n([M(a_1, \dots, a_n)]).$$

Assume that  $[M(a_1, \dots, a_n)] \in B_{\mathbf{d}_0}$  for some  $\mathbf{d}_0 \in [\mathbf{d}]$ , by (3.8)–(3.11), we have

$$\begin{aligned}
& [\mathcal{E}_3([\mathbf{d}])](M(a_1, \dots, a_n)) \\
&= [\bigoplus_{\mathbf{d}' \in [\mathbf{d}]} (\mathcal{E}_3^0(\mathbf{d}') \oplus \mathcal{E}_3^2(\mathbf{d}') \oplus \mathcal{E}_3^4(\mathbf{d}') \oplus \mathcal{E}_3^6(\mathbf{d}')) M(a_1, \dots, a_n)] \\
&= [(\mathcal{E}_3^0(\mathbf{d}_0) \oplus \mathcal{E}_3^2(\mathbf{d}_0) \oplus \mathcal{E}_3^4(\mathbf{d}_0) \oplus \mathcal{E}_3^6(\mathbf{d}_0)) M(a_1, \dots, a_n)] \\
&= \sum_{\substack{m=1, \\ a_m=0}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)] + \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)] \\
&\quad + \sum_{\substack{m=1, \\ a_m=4}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)] + \sum_{\substack{m=1, \\ a_m=6}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)].
\end{aligned}$$

It follows that

$$\begin{aligned}
& \gamma_n \circ [\mathcal{E}_3([\mathbf{d}])](M(a_1, \dots, a_n)) \\
&= \gamma_n \left( \sum_{\substack{m=1, \\ a_m=0}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)] \right) + \gamma_n \left( \sum_{\substack{m=1, \\ a_m=2}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)] \right) \\
&\quad + \gamma_n \left( \sum_{\substack{m=1, \\ a_m=4}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)] \right) + \gamma_n \left( \sum_{\substack{m=1, \\ a_m=6}}^n [M(a_1, \dots, a_{m-1}, a_m + 1, a_{m+1}, \dots, a_n)] \right) \\
&= \sum_{\substack{m=1, \\ a_m=0}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) + \sum_{\substack{m=1, \\ a_m=2}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) \\
&\quad + \sum_{\substack{m=1, \\ a_m=4}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) + \sum_{\substack{m=1, \\ a_m=6}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+2} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
& e_3 \circ \gamma_n([M(a_1, \dots, a_n)]) \\
&= e_3(v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) \\
&= \sum_{m=1}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes e_3 v_{a_m} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) \\
&= \sum_{\substack{m=1, \\ a_m=0}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+1} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) + \sum_{\substack{m=1, \\ a_m=2}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+1} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) \\
&\quad + \sum_{\substack{m=1, \\ a_m=4}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+1} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}) + \sum_{\substack{m=1, \\ a_m=6}}^n (v_{a_1} \otimes \dots \otimes v_{a_{m-1}} \otimes v_{a_m+1} \otimes v_{a_{m+1}} \otimes \dots \otimes v_{a_n}).
\end{aligned}$$

Hence  $\gamma_n \circ [\mathcal{E}_3([\mathbf{d}])](M(a_1, \dots, a_n)) = e_3 \circ \gamma_n([M(a_1, \dots, a_n)])$ .  $\square$

**Theorem 3.3.** For any  $1 \leq i \leq 3$  and  $2 \leq j \leq 3$ , let

$$\begin{aligned}
\mathcal{H}_i^+ &= \bigoplus_{\substack{[\mathbf{d}] \in \bar{\mathbf{D}}, \\ \text{sgn}(c_i(\mathbf{d}))=1 \text{ or } 0}} \mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n, & \mathcal{H}_i^- &= \bigoplus_{\substack{[\mathbf{d}] \in \bar{\mathbf{D}}, \\ \text{sgn}(c_i(\mathbf{d}))=-1}} \mathcal{H}_i^{\text{sgn}(c_i(\mathbf{d}))}([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n, \\
\mathcal{E}_1^+ &= \bigoplus_{[\mathbf{d}] \in \bar{\mathbf{D}}} \mathcal{E}_1^+([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n, & \mathcal{E}_1^- &= \bigoplus_{[\mathbf{d}] \in \bar{\mathbf{D}}} \mathcal{E}_1^-([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n, \\
\mathcal{F}_1^+ &= \bigoplus_{[\mathbf{d}] \in \bar{\mathbf{D}}} \mathcal{F}_1^+([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n, & \mathcal{F}_1^- &= \bigoplus_{[\mathbf{d}] \in \bar{\mathbf{D}}} \mathcal{F}_1^-([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n, \\
\mathcal{E}_j &= \bigoplus_{[\mathbf{d}] \in \bar{\mathbf{D}}} \mathcal{E}_j([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n, & \mathcal{F}_j &= \bigoplus_{[\mathbf{d}] \in \bar{\mathbf{D}}} \mathcal{F}_j([\mathbf{d}]) : \emptyset^n \rightarrow \emptyset^n.
\end{aligned}$$

Then we have the following results.

- (1) For any  $1 \leq i \leq 3$ , the action of  $h_i$  on  $V_{\text{sp}}^{\otimes n}$  can be categorified by a pair of exact functors  $(\mathcal{H}_i^+, \mathcal{H}_i^-)$ , which means that the following diagram commutes:

$$\begin{array}{ccc}
K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n} \\
[\mathcal{H}_i^+] - [\mathcal{H}_i^-] \downarrow & & \downarrow h_i \\
K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n}.
\end{array}$$

- (2) (a) The action of  $e_1$  on  $V_{\text{sp}}^{\otimes n}$  can be categorified by a pair of exact functors  $(\mathcal{E}_1^+, \mathcal{E}_1^-)$ , which means that the following diagram commutes:

$$\begin{array}{ccc}
K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n} \\
[\mathcal{E}_1^+] - [\mathcal{E}_1^-] \downarrow & & \downarrow e_1 \\
K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n}.
\end{array}$$

- (b) For any  $2 \leq i \leq 3$ , the action of  $e_i$  on  $V_{\text{sp}}^{\otimes n}$  can be categorified by the exact functor  $\mathcal{E}_i$ , which means that the following diagram commutes:

$$\begin{array}{ccc}
K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n} \\
[\mathcal{E}_i] \downarrow & & \downarrow e_i \\
K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n}.
\end{array}$$

- (3) (a) The action of  $f_1$  on  $V_{\text{sp}}^{\otimes n}$  can be categorified by a pair of exact functors  $(\mathcal{F}_1^+, \mathcal{F}_1^-)$ , which means that the following diagram commutes:

$$\begin{array}{ccc} K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n} \\ [\mathcal{F}_1^+] - [\mathcal{F}_1^-] \downarrow & & \downarrow f_1 \\ K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n}. \end{array}$$

- (b) For any  $2 \leq i \leq 3$ , the action of  $f_i$  on  $V_{\text{sp}}^{\otimes n}$  can be categorified by the exact functor  $\mathcal{F}_i$ , which means that the following diagram commutes:

$$\begin{array}{ccc} K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n} \\ [\mathcal{F}_i] \downarrow & & \downarrow f_i \\ K(\emptyset^n) & \xrightarrow{\gamma_n} & \mathbb{Z}V_{\text{sp}}^{\otimes n}. \end{array}$$

*Proof.* It is not difficult to check the diagrams are commutative by Proposition 3.2.  $\square$

Now we categorify defining relations of  $U(\mathfrak{so}(7, \mathbb{C}))$  as natural isomorphisms between some projective functors of  $\emptyset(\mathfrak{gl}_n)$ . Proposition 2.2 (5) plays an important role in the proof of the following results.

**Theorem 3.4.** (1)  $\mathcal{H}_i^+ \circ \mathcal{H}_j^+ \oplus \mathcal{H}_i^- \circ \mathcal{H}_j^- \oplus \mathcal{H}_j^+ \circ \mathcal{H}_i^- \oplus \mathcal{H}_j^- \circ \mathcal{H}_i^+ \cong \mathcal{H}_i^+ \circ \mathcal{H}_j^- \oplus \mathcal{H}_i^- \circ \mathcal{H}_j^+ \oplus \mathcal{H}_j^+ \circ \mathcal{H}_i^+ \oplus \mathcal{H}_j^- \circ \mathcal{H}_i^-$  for  $1 \leq i, j \leq 3$ .

- (2) (a)  $\mathcal{E}_1^+ \circ \mathcal{F}_1^+ \oplus \mathcal{E}_1^- \circ \mathcal{F}_1^- \oplus \mathcal{F}_1^+ \circ \mathcal{E}_1^- \oplus \mathcal{F}_1^- \circ \mathcal{E}_1^+ \oplus \mathcal{H}_1^- \cong \mathcal{E}_1^+ \circ \mathcal{F}_1^- \oplus \mathcal{E}_1^- \circ \mathcal{F}_1^+ \oplus \mathcal{F}_1^+ \circ \mathcal{E}_1^+ \oplus \mathcal{F}_1^- \circ \mathcal{E}_1^- \oplus \mathcal{H}_1^+$ ;  
 (b)  $\mathcal{E}_1^+ \circ \mathcal{F}_j \oplus \mathcal{F}_j \circ \mathcal{E}_1^- \cong \mathcal{E}_1^- \circ \mathcal{F}_j \oplus \mathcal{F}_j \circ \mathcal{E}_1^+$  for  $j = 2, 3$ ;  
 (c)  $\mathcal{E}_i \circ \mathcal{F}_1^+ \oplus \mathcal{F}_1^- \circ \mathcal{E}_i \cong \mathcal{E}_i \circ \mathcal{F}_1^- \oplus \mathcal{F}_1^+ \circ \mathcal{E}_i$  for  $i = 2, 3$ ;  
 (d)  $\mathcal{E}_i \circ \mathcal{F}_j \oplus \delta_{i,j} \mathcal{H}_i^- \cong \mathcal{F}_j \circ \mathcal{E}_i \oplus \delta_{i,j} \mathcal{H}_i^+$  for  $(i, j) = (2, 2), (2, 3), (3, 2)$  or  $(3, 3)$ .  
 (3) (a)  $\mathcal{H}_1^+ \circ \mathcal{E}_1^+ \oplus \mathcal{H}_1^- \circ \mathcal{E}_1^- \oplus \mathcal{E}_1^+ \circ \mathcal{H}_1^- \oplus \mathcal{E}_1^- \circ \mathcal{H}_1^+ \oplus (\mathcal{E}_1^-)^{\oplus 2} \cong \mathcal{H}_1^+ \circ \mathcal{E}_1^- \oplus \mathcal{H}_1^- \circ \mathcal{E}_1^+ \oplus \mathcal{E}_1^+ \circ \mathcal{H}_1^+ \oplus \mathcal{E}_1^- \circ \mathcal{H}_1^- \oplus (\mathcal{E}_1^+)^{\oplus 2}$ ;  
 (b)  $\mathcal{H}_i^+ \circ \mathcal{E}_1^+ \oplus \mathcal{H}_i^- \circ \mathcal{E}_1^- \oplus \mathcal{E}_1^+ \circ \mathcal{H}_i^- \oplus \mathcal{E}_1^- \circ \mathcal{H}_i^+ \oplus (\mathcal{E}_1^+)^{\oplus (-a_{i,1})} \cong \mathcal{H}_i^+ \circ \mathcal{E}_1^- \oplus \mathcal{H}_i^- \circ \mathcal{E}_1^+ \oplus \mathcal{E}_1^+ \circ \mathcal{H}_i^+ \oplus \mathcal{E}_1^- \circ \mathcal{H}_i^- \oplus (\mathcal{E}_1^-)^{\oplus (-a_{i,1})}$  for  $i = 2, 3$ ;  
 (c)  $\mathcal{H}_i^+ \circ \mathcal{E}_j \oplus \mathcal{E}_j \circ \mathcal{H}_i^- \oplus (\mathcal{E}_j)^{\oplus (-a_{i,j})} \cong \mathcal{H}_i^- \circ \mathcal{E}_j \oplus \mathcal{E}_j \circ \mathcal{H}_i^+ \oplus (\mathcal{E}_j)^{\oplus (-a_{i,j})}$  for  $(i, j) = (1, 2), (1, 3), (2, 3)$  or  $(3, 2)$ ;  
 (d)  $\mathcal{H}_i^+ \circ \mathcal{E}_i \oplus \mathcal{E}_i \circ \mathcal{H}_i^- \cong \mathcal{H}_i^- \circ \mathcal{E}_i \oplus \mathcal{E}_i \circ \mathcal{H}_i^+ \oplus \mathcal{E}_i^{\oplus 2}$  for  $i = 2, 3$ .  
 (4) (a)  $\mathcal{H}_1^+ \circ \mathcal{F}_1^+ \oplus \mathcal{H}_1^- \circ \mathcal{F}_1^- \oplus \mathcal{F}_1^+ \circ \mathcal{H}_1^- \oplus \mathcal{F}_1^- \circ \mathcal{H}_1^+ \oplus (\mathcal{F}_1^+)^{\oplus 2} \cong \mathcal{H}_1^+ \circ \mathcal{F}_1^- \oplus \mathcal{H}_1^- \circ \mathcal{F}_1^+ \oplus \mathcal{F}_1^+ \circ \mathcal{H}_1^+ \oplus \mathcal{F}_1^- \circ \mathcal{H}_1^- \oplus (\mathcal{F}_1^-)^{\oplus 2}$ ;  
 (b)  $\mathcal{H}_i^+ \circ \mathcal{F}_1^+ \oplus \mathcal{H}_i^- \circ \mathcal{F}_1^- \oplus \mathcal{F}_1^+ \circ \mathcal{H}_i^- \oplus \mathcal{F}_1^- \circ \mathcal{H}_i^+ \oplus (\mathcal{F}_1^-)^{\oplus (-a_{i,1})} \cong \mathcal{H}_i^+ \circ \mathcal{F}_1^- \oplus \mathcal{H}_i^- \circ \mathcal{F}_1^+ \oplus \mathcal{F}_1^+ \circ \mathcal{H}_i^+ \oplus \mathcal{F}_1^- \circ \mathcal{H}_i^- \oplus (\mathcal{F}_1^+)^{\oplus (-a_{i,1})}$  for  $i = 2, 3$ ;  
 (c)  $\mathcal{H}_i^+ \circ \mathcal{F}_j \oplus \mathcal{F}_j \circ \mathcal{H}_i^- \cong \mathcal{H}_i^- \circ \mathcal{F}_j \oplus \mathcal{F}_j \circ \mathcal{H}_i^+ \oplus (\mathcal{F}_j)^{\oplus (-a_{i,j})}$  for  $(i, j) = (1, 2), (1, 3), (2, 3)$  or  $(3, 2)$ ;  
 (d)  $\mathcal{H}_i^+ \circ \mathcal{F}_i \oplus \mathcal{F}_i \circ \mathcal{H}_i^- \oplus \mathcal{F}_i^{\oplus 2} \cong \mathcal{H}_i^- \circ \mathcal{F}_i \oplus \mathcal{F}_i \circ \mathcal{H}_i^+ \oplus \mathcal{F}_i^{\oplus 2}$  for  $i = 2, 3$ .  
 (5) (a)  $\mathcal{E}_1^+ \circ \mathcal{E}_1^+ \oplus \mathcal{E}_2 \oplus \mathcal{E}_1^- \circ \mathcal{E}_2 \oplus \mathcal{E}_2 \oplus \mathcal{E}_1^+ \circ \mathcal{E}_1^+ \oplus \mathcal{E}_2 \circ \mathcal{E}_1^- \oplus \mathcal{E}_1^- \oplus (\mathcal{E}_1^+ \circ \mathcal{E}_2 \circ \mathcal{E}_1^-)^{\oplus 2} \oplus (\mathcal{E}_1^- \circ \mathcal{E}_2 \circ \mathcal{E}_1^+)^{\oplus 2} \cong \mathcal{E}_1^+ \circ \mathcal{E}_1^- \oplus \mathcal{E}_2 \oplus \mathcal{E}_1^- \circ \mathcal{E}_2 \oplus \mathcal{E}_1^+ \circ \mathcal{E}_2 \oplus \mathcal{E}_2 \circ \mathcal{E}_1^- \oplus \mathcal{E}_2 \circ \mathcal{E}_1^- \oplus \mathcal{E}_1^+ \oplus (\mathcal{E}_1^+ \circ \mathcal{E}_2 \circ \mathcal{E}_1^+)^{\oplus 2} \oplus (\mathcal{E}_1^- \circ \mathcal{E}_2 \circ \mathcal{E}_1^-)^{\oplus 2}$ ;  
 (b)  $\mathcal{E}_1^+ \circ \mathcal{E}_3 \oplus \mathcal{E}_3 \circ \mathcal{E}_1^- \cong \mathcal{E}_1^- \circ \mathcal{E}_3 \oplus \mathcal{E}_3 \circ \mathcal{E}_1^+$ ;  
 (c)  $\mathcal{E}_2 \circ \mathcal{E}_2 \circ \mathcal{E}_1^+ \oplus \mathcal{E}_1^+ \circ \mathcal{E}_2 \oplus \mathcal{E}_2 \oplus (\mathcal{E}_2 \circ \mathcal{E}_1^- \circ \mathcal{E}_2)^{\oplus 2} \cong \mathcal{E}_2 \circ \mathcal{E}_2 \circ \mathcal{E}_1^- \oplus \mathcal{E}_1^- \circ \mathcal{E}_2 \oplus \mathcal{E}_2 \oplus (\mathcal{E}_2 \circ \mathcal{E}_1^+ \circ \mathcal{E}_2)^{\oplus 2}$ ;  
 (d)  $\mathcal{E}_2 \circ \mathcal{E}_2 \circ \mathcal{E}_3 \oplus \mathcal{E}_3 \circ \mathcal{E}_2 \oplus \mathcal{E}_2 \oplus \mathcal{E}_2 \cong (\mathcal{E}_2 \circ \mathcal{E}_3 \circ \mathcal{E}_2)^{\oplus 2}$ ;  
 (e)  $\mathcal{E}_3 \circ \mathcal{E}_3 \circ \mathcal{E}_3 \oplus \mathcal{E}_2 \oplus (\mathcal{E}_3 \circ \mathcal{E}_2 \circ \mathcal{E}_3 \circ \mathcal{E}_3)^{\oplus 3} \cong (\mathcal{E}_3 \circ \mathcal{E}_3 \circ \mathcal{E}_2 \circ \mathcal{E}_3)^{\oplus 3} \oplus \mathcal{E}_2 \circ \mathcal{E}_3 \circ \mathcal{E}_3 \circ \mathcal{E}_3$ .  
 (6) (a)  $\mathcal{F}_1^+ \circ \mathcal{F}_1^+ \oplus \mathcal{F}_2 \oplus \mathcal{F}_1^- \circ \mathcal{F}_2 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1^+ \circ \mathcal{F}_1^+ \oplus \mathcal{F}_2 \circ \mathcal{F}_1^- \oplus \mathcal{F}_1^- \oplus (\mathcal{F}_1^+ \circ \mathcal{F}_2 \circ \mathcal{F}_1^-)^{\oplus 2} \oplus (\mathcal{F}_1^- \circ \mathcal{F}_2 \circ \mathcal{F}_1^+)^{\oplus 2} \cong \mathcal{F}_1^+ \circ \mathcal{F}_1^- \oplus \mathcal{F}_2 \oplus \mathcal{F}_1^- \circ \mathcal{F}_2 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1^+ \circ \mathcal{F}_1^+ \oplus \mathcal{F}_2 \circ \mathcal{F}_1^- \oplus \mathcal{F}_1^- \oplus (\mathcal{F}_1^+ \circ \mathcal{F}_2 \circ \mathcal{F}_1^+)^{\oplus 2} \oplus (\mathcal{F}_1^- \circ \mathcal{F}_2 \circ \mathcal{F}_1^-)^{\oplus 2}$ ;  
 (b)  $\mathcal{F}_1^+ \circ \mathcal{F}_3 \oplus \mathcal{F}_3 \circ \mathcal{F}_1^- \cong \mathcal{F}_1^- \circ \mathcal{F}_3 \oplus \mathcal{F}_3 \circ \mathcal{F}_1^+$ ;  
 (c)  $\mathcal{F}_2 \circ \mathcal{F}_2 \circ \mathcal{F}_1^+ \oplus \mathcal{F}_1^+ \circ \mathcal{F}_2 \oplus \mathcal{F}_2 \oplus (\mathcal{F}_2 \circ \mathcal{F}_1^- \circ \mathcal{F}_2)^{\oplus 2} \cong \mathcal{F}_2 \circ \mathcal{F}_2 \circ \mathcal{F}_1^- \oplus \mathcal{F}_1^- \circ \mathcal{F}_2 \oplus \mathcal{F}_2 \oplus (\mathcal{F}_2 \circ \mathcal{F}_1^+ \circ \mathcal{F}_2)^{\oplus 2}$ ;  
 (d)  $\mathcal{F}_2 \circ \mathcal{F}_2 \circ \mathcal{F}_3 \oplus \mathcal{F}_3 \circ \mathcal{F}_2 \oplus \mathcal{F}_2 \oplus \mathcal{F}_2 \cong (\mathcal{F}_2 \circ \mathcal{F}_3 \circ \mathcal{F}_2)^{\oplus 2}$ ;

$$(e) \quad \mathcal{F}_3 \circ \mathcal{F}_3 \circ \mathcal{F}_3 \circ \mathcal{F}_2 \oplus (\mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_3 \circ \mathcal{F}_3)^{\oplus 3} \cong (\mathcal{F}_3 \circ \mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_3)^{\oplus 3} \oplus \mathcal{F}_2 \circ \mathcal{F}_3 \circ \mathcal{F}_3 \circ \mathcal{F}_3.$$

*Proof.* We check Theorem 3.4 in some cases, while the remaining cases can be verified similarly.

(1) By Proposition 2.2 (5), we only need to check that

$$(3.22) \quad \begin{aligned} & [\mathcal{H}_i^+] \circ [\mathcal{H}_j^+] + [\mathcal{H}_i^-] \circ [\mathcal{H}_j^-] + [\mathcal{H}_j^+] \circ [\mathcal{H}_i^-] + [\mathcal{H}_j^-] \circ [\mathcal{H}_i^+] \\ &= [\mathcal{H}_i^+] \circ [\mathcal{H}_j^-] + [\mathcal{H}_i^-] \circ [\mathcal{H}_j^+] + [\mathcal{H}_j^+] \circ [\mathcal{H}_i^+] + [\mathcal{H}_j^-] \circ [\mathcal{H}_i^-]. \end{aligned}$$

Indeed, it is easy to see that  $h_i h_j \gamma_n = h_j h_i \gamma_n$ . By Theorem 3.3 (1), we have

$$\begin{aligned} h_i h_j \gamma_n &= \gamma_n \circ ([\mathcal{H}_i^+] - [\mathcal{H}_i^-]) \circ ([\mathcal{H}_j^+] - [\mathcal{H}_j^-]), \\ h_j h_i \gamma_n &= \gamma_n \circ ([\mathcal{H}_j^+] - [\mathcal{H}_j^-]) \circ ([\mathcal{H}_i^+] - [\mathcal{H}_i^-]). \end{aligned}$$

Therefore,

$$(3.23) \quad ([\mathcal{H}_i^+] - [\mathcal{H}_i^-]) \circ ([\mathcal{H}_j^+] - [\mathcal{H}_j^-]) = ([\mathcal{H}_j^+] - [\mathcal{H}_j^-]) \circ ([\mathcal{H}_i^+] - [\mathcal{H}_i^-]).$$

We immediately get (3.22) by expanding (3.23).

(2) (a) By Proposition 2.2 (5), it is enough to check that

$$(3.24) \quad \begin{aligned} & [\mathcal{E}_1^+] \circ [\mathcal{F}_1^+] + [\mathcal{E}_1^-] \circ [\mathcal{F}_1^-] + [\mathcal{F}_1^+] \circ [\mathcal{E}_1^-] + [\mathcal{F}_1^-] \circ [\mathcal{E}_1^+] + [\mathcal{H}_1^-] \\ &= [\mathcal{E}_1^+] \circ [\mathcal{F}_1^-] + [\mathcal{E}_1^-] \circ [\mathcal{F}_1^+] + [\mathcal{F}_1^+] \circ [\mathcal{E}_1^+] + [\mathcal{F}_1^-] \circ [\mathcal{E}_1^-] + [\mathcal{H}_1^-]. \end{aligned}$$

Indeed, noting that

- (i)  $(e_1 f_1 - f_1 e_1) \gamma_n = h_1 \gamma_n$ ,
- (ii)  $(e_1 f_1 - f_1 e_1) \gamma_n = \gamma_n \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) - ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-])$ ,
- (iii)  $h_1 \gamma_n = \gamma_n \circ ([\mathcal{H}_1^+] - [\mathcal{H}_1^-])$ ,

where (ii) and (iii) follow from Theorem 3.3 (1), (2)(a) and (3)(a), we have

$$(3.25) \quad ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) - ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) = [\mathcal{H}_1^+] - [\mathcal{H}_1^-],$$

(3.24) is obtained by expanding (3.25).

(3) (a) By Proposition 2.2 (5), it suffices to check that

$$(3.26) \quad \begin{aligned} & [\mathcal{H}_1^+] \circ [\mathcal{E}_1^+] + [\mathcal{H}_1^-] \circ [\mathcal{E}_1^-] + [\mathcal{E}_1^+] \circ [\mathcal{H}_1^-] + [\mathcal{E}_1^-] \circ [\mathcal{H}_1^+] + 2[\mathcal{E}_1^-] \\ &= [\mathcal{H}_1^+] \circ [\mathcal{E}_1^-] + [\mathcal{H}_1^-] \circ [\mathcal{E}_1^+] + [\mathcal{E}_1^+] \circ [\mathcal{H}_1^+] + [\mathcal{E}_1^-] \circ [\mathcal{H}_1^-] + 2[\mathcal{E}_1^-]. \end{aligned}$$

Indeed, noting that

- (i)  $(h_1 e_1 - e_1 h_1) \gamma_n = 2e_1 \gamma_n$ ,
- (ii)  $(h_1 e_1 - e_1 h_1) \gamma_n = \gamma_n \circ ([\mathcal{H}_1^+] - [\mathcal{H}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) - ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{H}_1^+] - [\mathcal{H}_1^-])$ ,
- (iii)  $2e_1 \gamma_n = 2\gamma_n \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-])$ ,

where (ii) and (iii) can be seen from Theorem 3.3 (1) and (2)(a), we have

$$(3.27) \quad ([\mathcal{H}_1^+] - [\mathcal{H}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) - ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{H}_1^+] - [\mathcal{H}_1^-]) = 2([\mathcal{E}_1^+] - [\mathcal{E}_1^-]),$$

(3.26) follows from expanding (3.27).

(4) (b) We only give the proof of the case  $(i, j) = (2, 1)$ . By Proposition 2.2 (5), it is sufficient to check that

$$(3.28) \quad \begin{aligned} & [\mathcal{H}_2^+] \circ [\mathcal{F}_1^+] + [\mathcal{H}_2^-] \circ [\mathcal{F}_1^-] + [\mathcal{F}_1^+] \circ [\mathcal{H}_2^-] + [\mathcal{F}_1^-] \circ [\mathcal{H}_2^+] + [\mathcal{F}_1^-] \\ &= [\mathcal{H}_2^+] \circ [\mathcal{F}_1^-] + [\mathcal{H}_2^-] \circ [\mathcal{F}_1^+] + [\mathcal{F}_1^+] \circ [\mathcal{H}_2^+] + [\mathcal{F}_1^-] \circ [\mathcal{H}_2^-] + [\mathcal{F}_1^+]. \end{aligned}$$

Indeed, noting that

- (i)  $(h_2 f_1 - f_1 h_2) \gamma_n = f_1 \gamma_n$ ,
- (ii)  $(h_2 f_1 - f_1 h_2) \gamma_n = \gamma_n \circ [(\mathcal{H}_2^+ - [\mathcal{H}_2^-]) \circ ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) - ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) \circ ([\mathcal{H}_2^+ - [\mathcal{H}_2^-])]$ ,
- (iii)  $f_1 \gamma_n = \gamma_n \circ ([\mathcal{F}_1^+] - [\mathcal{F}_1^-])$ ,

where (ii) and (iii) follow from Theorem 3.3 (1) and (3)(a), we have

$$(3.29) \quad ([\mathcal{H}_2^+] - [\mathcal{H}_2^-]) \circ ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) - ([\mathcal{F}_1^+] - [\mathcal{F}_1^-]) \circ ([\mathcal{H}_2^+] - [\mathcal{H}_2^-]) = [\mathcal{F}_1^+] - [\mathcal{F}_1^-],$$

We obtain (3.28) by expanding (3.29).

(5) (a) By Proposition 2.2 (5), it is equivalent to check

$$(3.30) \quad \begin{aligned} & [\mathcal{E}_1^+] \circ [\mathcal{E}_1^+] \circ [\mathcal{E}_2] + [\mathcal{E}_1^-] \circ [\mathcal{E}_1^-] \circ [\mathcal{E}_2] + [\mathcal{E}_2] \circ [\mathcal{E}_1^+] \circ [\mathcal{E}_1^+] \\ & + [\mathcal{E}_2] \circ [\mathcal{E}_1^-] \circ [\mathcal{E}_1^-] + 2[\mathcal{E}_1^+] \circ [\mathcal{E}_2] \circ [\mathcal{E}_1^-] + 2[\mathcal{E}_1^-] \circ [\mathcal{E}_2] \circ [\mathcal{E}_1^+] \\ = & [\mathcal{E}_1^+] \circ [\mathcal{E}_1^-] \circ [\mathcal{E}_2] + [\mathcal{E}_1^-] \circ [\mathcal{E}_1^+] \circ [\mathcal{E}_2] + [\mathcal{E}_2] \circ [\mathcal{E}_1^+] \circ [\mathcal{E}_1^-] \\ & + [\mathcal{E}_2] \circ [\mathcal{E}_1^-] \circ [\mathcal{E}_1^+] + 2[\mathcal{E}_1^+] \circ [\mathcal{E}_2] \circ [\mathcal{E}_1^+] + 2([\mathcal{E}_1^-] \circ [\mathcal{E}_2] \circ [\mathcal{E}_1^-]). \end{aligned}$$

Indeed, noting that

- (i)  $(e_1^2 e_2 - 2e_1 e_2 e_1 + e_2 e_1^2) \gamma_n = 0$ ,
- (ii) by Theorem 3.3 (2) (a) and (b),  $(e_1^2 e_2 - 2e_1 e_2 e_1 + e_2 e_1^2) \gamma_n = \gamma_n \circ [([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ [\mathcal{E}_2] - 2([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ [\mathcal{E}_2] \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) + [\mathcal{E}_2] \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-])]$ ,  
we have (3.31)

$$([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ [\mathcal{E}_2] - 2([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ [\mathcal{E}_2] \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) + [\mathcal{E}_2] \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) \circ ([\mathcal{E}_1^+] - [\mathcal{E}_1^-]) = 0,$$

which is (3.30) after expanding.

(6) (e) By Proposition 2.2 (5), it is equivalent to verify

$$[\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_2] + 3[\mathcal{F}_3] \circ [\mathcal{F}_2] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] = 3[\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_2] \circ [\mathcal{F}_3] + [\mathcal{F}_2] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3].$$

By Theorem 3.3 (3) (b), we have

$$\begin{aligned} & \gamma_n \circ ([\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_2] + 3[\mathcal{F}_3] \circ [\mathcal{F}_2] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3]) \\ & = (f_3^3 f_2 + 3f_3 f_2 f_3^2) \circ \gamma_n = (3f_3^2 f_2 f_3 + f_2 f_3^3) \circ \gamma_n \\ & = \gamma_n \circ (3[\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_2] \circ [\mathcal{F}_3] + [\mathcal{F}_2] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3]). \end{aligned}$$

Thus  $[\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_2] + 3[\mathcal{F}_3] \circ [\mathcal{F}_2] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] = 3[\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_2] \circ [\mathcal{F}_3] + [\mathcal{F}_2] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3] \circ [\mathcal{F}_3]$   
since  $\gamma_n$  is an abelian group isomorphism.

□

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